# Hörmander's theorem for McKean-Vlasov SDEs

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We are here interested in studying the smoothing properties of some non-linear diffusion processes given by the unique strong solution to the following Stochastic Differential Equations (SDEs for short) with dynamics :

$$\begin{cases} X^{t,\xi}_s = \xi + \int_t^s A_0(X^{\xi}_u, m(u;t,\mu))\,\mathrm{du} + \sum_{\ell=1}^{\mathrm{q}} \int_{\mathrm{t}}^{\mathrm{s}} \mathrm{V}_\ell(\mathrm{X}^{\mathrm{t},\xi}_{\mathrm{u}}, \mathrm{m}(\mathrm{u};\mathrm{t},\mu))\,\mathrm{dB}^\ell_{\mathrm{u}}, \\ m(s;t,\mu) = \mathrm{Law}(X^{t,\xi}_s), \end{cases} \tag{1}$$

driven by a $q$-dimensional $B = (B^1, \cdots, B^q)$ Brownian motion, starting from the $\mathbb{R}^d$-valued random variable $\xi$ independent of $B$ and with law $\mu \in \mathcal{P}(\mathbb{R}^d)$.

In the uniform elliptic setting, the analysis of the smoothing properties of the density of $X^{t,\xi}_s$ and of the corresponding semigroup $\mathcal{P}_{s,t}\phi(\mu) = \phi(m(s;t,\mu))$ has received considerable attention in several recent works [1], [2], [3] and [4]. However, much less is known in the hypoelliptic setting.

Under an ad-hoc Hörmander's condition on the vector fields $V_0, \cdots, V_q : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$, it is shown that the density of $X^{t,\xi}_s$ exists and is smooth with respect to the initial data $t, \mu$. We also prove that the map $(t,\mu) \mapsto \mathcal{P}_{s,t}\phi(\mu)$ is the unique solution to a backward Kolmogorov PDE defined on the strip $[0,s] \times \mathcal{P}_q(\mathbb{R}^d)$, for any $s > 0$ and some $q \geq 2$ even if $\phi$ is not smooth. Some new quantitative weak propagation of chaos results for the mean-field approximation of (1) by the corresponding system of interacting particles are eventually established.

## Références

[1] D. Crisan, E. McMurray, *Smoothing Properties of McKean-Vlasov SDEs*, Probability Theory and Related Fields, volume 171, 2018, pages 97-148.

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[3] P.-E. Chaudru de Raynal, N. Frikha, *Well-posedness for some non-linear SDEs and related PDE on the Wasserstein space*, Journal de Mathématiques Pures et Appliquées, volume 159, 2022, pages 1-167.

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